The Operator Product Expansion

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6/10/2020

QFT 3 - Final Presentation

Outline

The Big Picture

The Big Picture

We've spent much of this quarter learning to deal with divergences, typically those of an ultraviolet nature. These divergences led to shifting in mass parameters and coupling constants through the process of renormalization, but were otherwise rather innocuous.

Renormalization gave us a great deal of freedom in that we were free to define our theory at any arbitrary momentum scale. But in order to describe the the correct physical theory, the renormalized Green's functions must be identical to the bare Green functions, up to the field strength renormalization.

$$G_R^{(n)}(\mu; x_1, ..., x_n) = Z^{-n/2} G_b^{(n)}(x_1, ..., x_n)$$

The Big Picture

We then asked, how would our Green's functions change if we looked at a different renormalization scale? This led to the Callen-Symanzik (renormalization group) equations. Solving these, we calculated how coupling constants and fields evolve with momentum scale.

Until now, we've utilized correlation functions to perform this analysis. But we will see that it is particularly useful to study the renormalization flow at the level of the operators themselves. But to do so, we will need to deal with the non-locality of operator products. This is where we will meet the Operator Product Expansion (OPE).

The Big Picture

After motivating the OPE, we will see that all of the interesting behavior at high energy (short distances) depends only on Wilson coefficents. These can be calculated once and then used for *any* process since they do not depend on any external fields that appear in a Green's function.

We will then use the OPE to look at QCD renormalization of the weak interactions. Since QCD has asymptotic freedom, we will see that the OPE allows us to determine the renormalization corrections in terms of the anomalous dimensions of the operators.

Renormalization Group Refresher/Setup

The bare and renormalized Green's functions are related by

$$G_R^{(n)}(\mu; x_1, ..., x_n) = Z^{-n/2} G_b^{(n)}(x_1, ..., x_n) \tag{1}$$

When varying the renormalization scale μ, we must also change Z and the coupling constants (here we'll use λ) in order for these Green's functions to represent the same physical theory. This can be written generally as

$$\frac{dG_R^{(n)}}{d\mu} = \frac{\partial G_R^{(n)}}{\partial \mu} + \frac{\partial G_R^{(n)}}{\partial \lambda}\frac{\partial \lambda}{\partial \mu} \tag{2}$$

By taking the derivative of the left hand side of (1) with respect to μ, and using the fact that the bare Green's function does not depend on the renormalization scale, we find

$$\frac{dG_R^{(n)}}{d\mu} = -\frac{n}{2Z}\frac{\partial Z}{\partial \mu}G_R^{(n)}. \tag{3}$$

Combining these results, we end up with the renormalization group equation

$$\left[\mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial \lambda} + n\gamma\right]G_R^{(n)} = 0, \tag{4}$$

where $\beta \equiv \mu\frac{\partial \lambda}{\partial \mu}$ and $\gamma \equiv \frac{\mu}{2Z}\frac{\partial Z}{\partial \mu}$ control how the coupling constant and field ϕ vary with μ respectively.

Correlation functions containing composite operators (operators evaluated at the same spacetime point) must also be renormalized. $\mathcal{O}_b = Z_{\mathcal{O}}\mathcal{O}_R$.

Thus, we may write the relationship between the bare and renormalized Green's functions as

$$G_R^{(n;1)}(\mu; x_1, ..., x_n; y) = Z^{-n/2} Z_{\mathcal{O}}^{-1} G_b^{(n;1)}(x_1, ..., x_n; y), \qquad (5)$$

where

$$G_R^{(n;1)}(\mu; x_1, ..., x_n; y) = \langle \phi(x_1)...\phi(x_n)\mathcal{O}(y)\rangle. \qquad (6)$$

By the same treatment as before, requiring the bare correlation function to be independent of the renormalization scale μ, we end up with the following equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma + \gamma_{\mathcal{O}}\right] G_R^{(n;1)} = 0, \tag{7}$$

where the anomalous dimension of the composite operator is defined as

$$\gamma_{\mathcal{O}} \equiv \frac{\mu}{Z_{\mathcal{O}}}\frac{\partial Z_{\mathcal{O}}}{\partial\mu} \tag{8}$$

It would be nice to study the renormalization flow at the level of the operators. But the behavior of composite operators is unwieldy $(A_1(x_1)A_2(x_2)...A_n(x_n)$ is singular when the arguments coincide $(x_1 = x_2... = x_n))$. For example, consider the humble free-field propagator

$$\langle 0| T\phi(x)\phi(y)|\rangle = D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}. \quad (9)$$

In the limit of $x \to y$ (forming a composite operator), this is divergent! To make our lives easier, we use the OPE to stick all the difficult behavior inside the Wilson coefficients.

Introducing the Operator Product Expansion

OPE

Let's say we're looking at a process that involves two operators, $\mathcal{O}_1$ and $\mathcal{O}_2$, which act on points separated by a small distance x. Let's also imagine that there are external physical states located much further away $\phi(y_i)$. The amplitude for this process can be calculated from the Green's function (where I'm writing all products of operators as time ordered)

$$G_{12}(x; , y_1...y_m) = \langle \mathcal{O}_1(x)\mathcal{O}_2(0)\phi(y_1)...\phi(y_m)\rangle. \tag{10}$$

In a correlation function like $\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\phi(y_1)...\phi(y_m)\rangle$, since the local operators $\phi(y_i)$ are far away from $0, x$, in the limit of $x \to 0$, the operator product looks like a single local operator. It can, in fact, produce the most general local disturbance. Thus, we can expand it in terms of the basis which contains all possible local operators at 0.

OPE

The operator product expansion (OPE) states that the product of local operators evaluated at different points, in the limit that those points approach each other, can be written as a sum over composite (local) operators:

$$\lim_{x \to 0} \mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_n C_n(x)\mathcal{O}_n(0). \tag{11}$$

This is useful because it holds at the level of operators; the OPE is independent of external states. In other words, regardless of any fields that might appear in a Green's function, once you have calculated a OPE once, you may then use it to calculate matrix elements for *any* process.

Callen-Symanzik Equation for Wilson Coefficients

Wilson Coefficients

Using the OPE, our Green's function can be expanded as

$$G_{12}(x; y_1, ... y_m) = \sum_n C_{12}^n(x) G_n(y_1, ... y_m),$$ (12)

where

$$G_n(y_1, ... y_m) = \langle \mathcal{O}_n(0) \phi(y_1) ... \phi(y_m) \rangle.$$ (13)

Notice that, indeed, all of the dependence on x is in the Wilson coefficients (the $C(x)$'s). But these, in turn, only depend on the operators themselves since

$$\lim_{x \to 0} \mathcal{O}_1(x) \mathcal{O}_2(0) = \sum_n C_n(x) \mathcal{O}_n(0).$$ (14)

Wilson Coefficients

We expect the Wilson coefficients to depend on the renormalization scale. So to determine this dependence, we write the RG equation for the renormalized Green's function

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma + \gamma_1 + \gamma_2\right] G_{12}^{(n)} = 0. \tag{15}$$

The RG equation for the Green functions G_n that show up on the right side of (12) give us

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma + \gamma_n\right] G_n = 0, \tag{16}$$

where γ_n is the anomalous dimension of the operator $\mathcal{O}_n$.

Wilson Coefficients

In order to satisfy

$$G_{12}(x; y_1, ...y_m) = \sum_n C_{12}^n(x) G_n(y_1, ...y_m),$$

the Wilson coefficients must obey the equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma_1 + \gamma_2 - \gamma_n\right] C_{12}^{(n)} = 0. \tag{17}$$

This shows that our previous assumption of the dependence of the coefficients on μ to be valid. Most importantly, this shows that the Wilson coefficients only depend on the anomalous dimensions of the operators, but not on the specific correlation function! Thus the number of fields ϕ and their anomalous dimension does not come into play.

x-dependence of Wilson Coefficients

Let's say the dimensions of the operators $\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_n$ are d_1, d_2, and d_n respectively. Then we may write the Wilson coefficients as

$$C_{12}^{(n)}(x; \mu) = \frac{1}{|x|^{d_1+d_2-d_n}} \tilde{C}_{12}^n(\mu|x|), \tag{18}$$

where $\tilde{C}_{12}^n$ is dimensionless. To find the structure of the coefficient, we introduce the running coupling $\overline{\lambda}(1/|x|)$, defined by

$$\frac{d\overline{\lambda}(p, \lambda)}{d\ln(p/\mu)} = \beta(\overline{\lambda}), \qquad \overline{\lambda}(\mu, \lambda) = \lambda$$

which lets us write the solution as

$$C_{12}^{(n)}(x; \mu) = \frac{\mathcal{C}_{12}^{(n)}\left((\overline{\lambda}(1/|x|))\right)}{|x|^{d_1+d_2-d_n}}$$
$$\times \exp\left[\int_{1/|x|}^{\mu} \frac{dp'}{p'}[\gamma_n(\overline{\lambda}(p')) - \gamma_{\mathcal{O}_1}(\overline{\lambda}(p')) - \gamma_{\mathcal{O}_2}(\overline{\lambda}(p'))]\right] \tag{19}$$

where $\mathcal{C}_{12}^{(n)}$ is a perturbative function of the running coupling.

x-dependence of Wilson Coefficients

$$C_{12}^{(n)}(x;\mu) = \frac{\mathcal{C}_{12}^{(n)}((\overline{\lambda}(1/|x|)))}{|x|^{d_1+d_2-d_n}}$$

$$\times \exp\Big[\int_{1/|x|}^{\mu}\frac{dp'}{p'}[\gamma_n(\overline{\lambda}(p')) - \gamma_{\mathcal{O}_1}(\overline{\lambda}(p')) - \gamma_{\mathcal{O}_2}(\overline{\lambda}(p'))]\Big]$$

We see that the short distance behavior is singular if $d_n < d_1 + d_2$. On the other hand, if $d_n > d_1 + d_2$, the contribution goes to zero so we do not need to consider operators of this kind in the OPE.

In class, we saw that QCD was asymptotically free. That is, the coupling goes to zero at short distances. Let's examine the Wilson coefficients further in this case.

QCD Wilson Coefficients

To first order, any anomalous dimension can be written as
$\gamma_{\mathcal{O}} \equiv -a_{\mathcal{O}} \frac{g^2}{(4\pi)^2}$ for some numerical constant $a_{\mathcal{O}}$. Thus we have

$$\gamma_d - \gamma_1 - \gamma_2 = (a_1 + a_2 - a_n)\frac{\alpha_s}{(4\pi)}. \tag{20}$$

At one loop, the running coupling α_s is simply

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(\frac{Q^2}{\Lambda_{\mathrm{QCD}}^2})} \tag{21}$$

where β_0 is the coefficient of the first term in the Taylor expansion of the QCD β function and Q is the momentum scale of the given process.

QCD Wilson Coefficients

This leads us to

$$C_{12}^{(n)}(x; \mu) = \frac{C_{12}^{(n)}\big(\overline{g}(1/|x|)\big)}{|x|^{d_1+d_2-d_n}} \left[\frac{\ln(1/|x|^2 \Lambda_{\mathrm{QCD}}^2)}{\ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)} \right]^{\frac{a_n - a_1 - a_2}{2\beta_0}}. \qquad (22)$$

In particular, note that when $d_n = d_1 + d_2$, the main source of the $|x|$ dependence is the powers of the logarithmic terms.

Operator Mixing

It is possible for quantum corrections to mix operators under the evolution of the renormalization scale. In that case, we must consider the more general form of the Callen-Symanzik equation for the Wilson Coefficients

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma_1 + \gamma_2\right]C_{12}^j - \sum_i \gamma_{ij}C_{12}^i = 0, \tag{23}$$

where γ_{ij} is now a matrix.

An example: QCD corrections to weak decays

Overview - QCD corrections to weak decays

We will see that the matrix element for a QCD process mediated by an operator $\mathcal{O}(x)$ will receive corrections which depend on the momentum scale of the process and the renormalization scale. These corrections will depend on the anomalous dimension of the operator which takes the form

$$\gamma_{\mathcal{O}} = -a_{\mathcal{O}} \frac{g^2}{(4\pi)^2}. \tag{24}$$

After solving the Callen-Symanzik equation, we will find the QCD renormaliztion factor

$$\left(\frac{\ln(\mu^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)} \right)^{a_{\mathcal{O}}/2b_0}, \tag{25}$$

where Q is the momentum scale of the process mediated by $\mathcal{O}$, μ is the renormalization scale used to define the operator normalization, and b_0 is the coefficient of the QCD β function $b_0 = 11 - \frac{2n_f}{3}$.

At low energy, the weak interaction between quarks and leptons in the standard model can be described by Fermi's 4-fermion theory. The interaction term in the Lagrangian is

$$\mathcal{L}_{int} \approx \frac{4G_F}{\sqrt{2}} J_L^\mu(0) J_L^{\nu\dagger}(0) + h.c., \tag{26}$$

where J_L^μ is the left handed charged current. In the full electroweak theory, the product of currents would be non-local and the Lagrangian would contain instead $J_L^\mu(0) D_{\mu\nu}(0,x) J_L^{\nu\dagger}(x)$. Using the OPE we will study how the non-local electroweak operators get replaced by local ones.

The Trivial Cases - QCD corrections to weak decays

Since leptons do not couple directly to gluons at one loop, there will be no QCD corrections to purely leptonic weak interactions.

For a semi-leptonic weak interaction (involving both a leptonic current and a quark current), again the leptons don't receive corrections to first order. Moreover, the quark current has an anomalous dimension of zero so it is not affected by strong interactions.

The only non-trivial case is non-leptonic weak interactions, i.e. weak interactions between quark currents. Let's look at these now.

Strange Decay - QCD corrections to weak decays

Let's look at the weak decay of the strange quark. In the standard model, the interactions between charged currents containing leptonic and quark terms has the form

$$\mathcal{L}_{\text{int}} = \frac{g^2}{2} J_L^\mu D_{\mu\nu}(0, x) J_L^{\nu\dagger}(x) + \text{h.c.}, \tag{27}$$

where $D_{\mu\nu}$ is the propagator of the W boson. At low momenta, we can make the approximation

$$\frac{1}{k^2 - m_W^2} \to \frac{-1}{m_W^2}, \tag{28}$$

in which case, the weak interaction becomes an effective local vertex (Fermi theory). We will find the consequences of the original composite interaction by working with this local operator.

In Fermi theory, we have the coupling $(\overline{d}_L \gamma^\mu u_L)(\overline{u}_L \gamma_\mu s_L)$. So let's then look at the OPE of the product of the currents

$$A_1^\mu(x) \equiv \overline{d}_L \gamma^\mu u_L, \qquad A_2^\mu(0) = \overline{u}_L \gamma_\mu s_L. \qquad (29)$$

These will be expanded on the operators

$$\mathcal{O}_1 \equiv (\overline{d}_L \gamma^\mu u_L)(\overline{u}_L \gamma_\mu s_L), \qquad \mathcal{O}_2 \equiv (\overline{d}_L \gamma^\mu s_L)(\overline{u}_L \gamma_\mu u_L). \qquad (30)$$

Note that $d_1 + d_2 = d_n$, so the x dependence of the Wilson coefficients comes entirely from the logarithms at one loop, as we saw previously.

Strange Decay - QCD corrections to weak decays

To determine the Wilson coefficients $C_{12}^{(n)}$ for the operators $\mathcal{O}_n$, we first need to calculate the anomalous dimensions of all the involved operators. Since A_1 and A_2 are conserved, their anomalous dimension is zero $\gamma_{A_1} = \gamma_{A_2} = 0$! This is a consequence of the Ward identity and can be verified easily. As we saw in class, a conserved currents imply that the corresponding charges ($Q = \int d^3x j^0$) form a Lie algebra

$$[Q_a, Q_b] = f_{ab}^c Q_C, \tag{31}$$

which by dimensional analysis tells us that conserved currents never acquire an anomalous dimension.

To get the anomalous dimension of the operators $\mathcal{O}_1, \mathcal{O}_2$, we need to calculate all the order g^2 corrections. For $\mathcal{O}_1$, these come from the diagrams
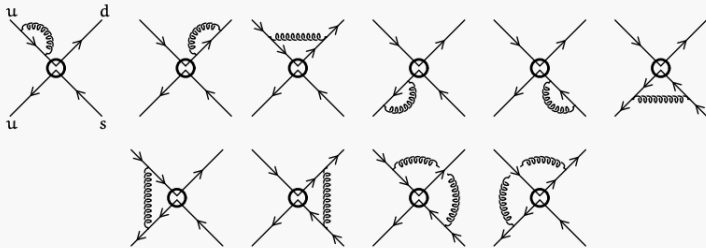

Figure 7.1: Order-g^2 QCD corrections to the operator $\mathcal{O}_1$.

Strange Decay - QCD corrections to weak decays

The calculation of the previous diagrams is not particularly important. See Gelis 7.3 for the details. The takeaway is that the operators mix, causing the anomalous dimensions for the operators to form a non-diagonal matrix

$$\gamma_{ij} = \frac{g^2}{(4\pi)^2} \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix} \tag{32}$$

Thus we must solve a coupled Callen-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda}\right] C_{12}^j - \sum_i \gamma_{ij} C_{12}^i = 0, \tag{33}$$

by rotating to a diagonal basis. This is satisfied for operators

$$\mathcal{O}_{1/2} \equiv \frac{1}{2}[\mathcal{O}_1 - \mathcal{O}_2], \quad \mathcal{O}_{3/2} = \frac{1}{2}[\mathcal{O}_1 + \mathcal{O}_2]. \tag{34}$$

Strange Decay - QCD corrections to weak decays

The eigenvalues of the matrix γ_{ij} are

$$\gamma_{1/2} = -8\frac{g^2}{(4\pi)^2}, \qquad \gamma_{3/2} = 4\frac{g^2}{(4\pi)^2}. \tag{35}$$

To first order then, we find the values for the Wilson coefficients at a distance scale of $x \approx m_w^{-1}$

$$
\begin{aligned}
C_{12}^{1/2}(m_w^{-1}; \mu) &= \Big[\frac{\ln(m_w^2/\Lambda_{\mathrm{QCD}}^2)}{\ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)}\Big]^{\frac{4}{\beta_0}} \\
C_{12}^{3/2}(m_w^{-1}; \mu) &= \Big[\frac{\ln(m_w^2/\Lambda_{\mathrm{QCD}}^2)}{\ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)}\Big]^{-\frac{2}{\beta_0}}.
\end{aligned}
\tag{36}
$$

The superscripts on these operators refer to their isospin quantum numbers. In turns out that $\mathcal{O}_{1/2}$ can mediate processes that change the isospin by $1/2$ but not by $3/2$.

Kaon Decays - QCD corrections to weak decays

Now that we finally have the QCD corrections to strange quark decays, let's quickly see how this applies to the K meson (the lightest hadron with an s quark). The process $K^0 \to \pi^+\pi^-$ changes the isospin by $1/2$. The process $K^+ \to \pi^+\pi^0$ changes isospin by $3/2$. Experimentally, the former occurs at a much higher rate.

For $\Lambda_{QCD} \approx 150$ MeV, $\mu = m_K$, we find the operator $\mathcal{O}_{1/2}$ receives an enhancement whereas the operator $\mathcal{O}_{3/2}$ receives a suppression. Thus the weak decay of the s quark favors isospin $1/2$ processes by a factor of ≈ 3. Experimentally, the relative difference bewtween the processes is ≈ 20. Getting the exact rates requires non-perturbative QCD, but the power of the OPE is that we were able to make this qualitative prediction using only perturbative methods.

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